1.5 Chapter 05.03 Newton's Divided Difference Interpolation

After reading this chapter, you should be able to:
1. derive Newton's divided difference method of interpolation,
2. apply Newton's divided difference method of interpolation, and
3. apply Newton's divided difference method interpolants to find derivatives and integrals.

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We will discuss Newton's divided difference polynomial method in this chapter.

Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.

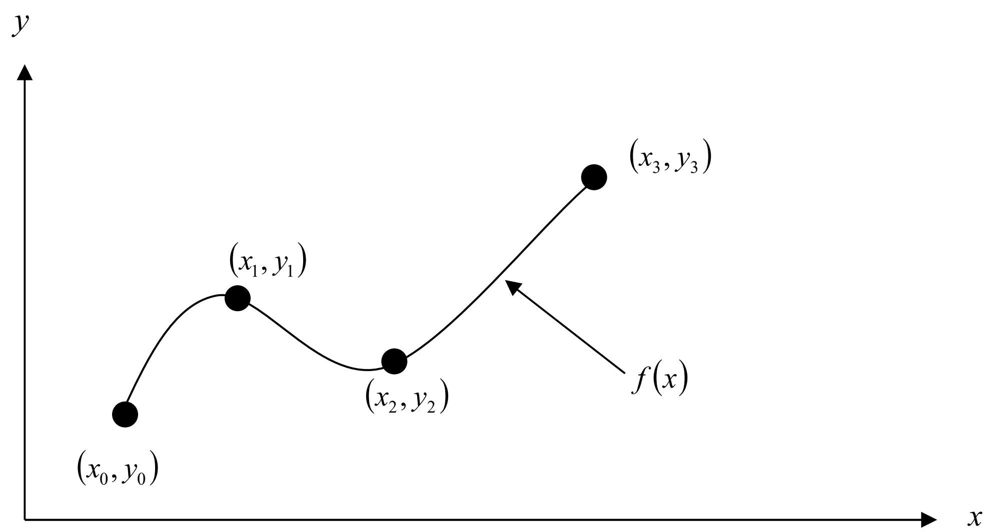


Figure 1 Interpolation of discrete data.

Linear Interpolation

Given (x_0, y_0) and (x_1, y_1), fit a linear interpolant through the data. Noting $y = f(x)$ and $y_1 = f(x_1)$, assume the linear interpolant $f_1(x)$ is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$
$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

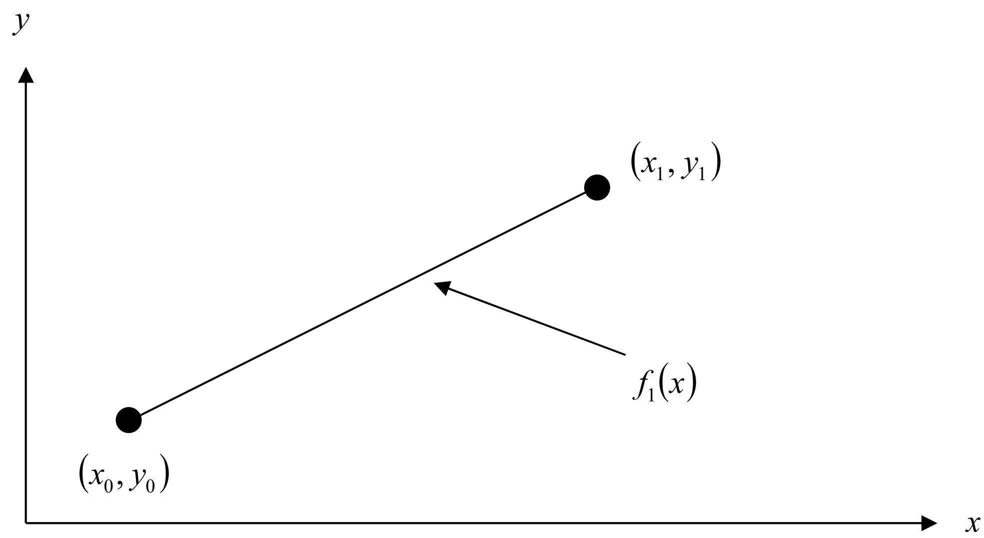


Figure 2 Linear interpolation.

Example 1
The upward velocity of a rocket is given as a function of time in Table 1 (Figure 3).

Table 1 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using first order polynomial interpolation by Newton's divided difference polynomial method.

Solution
For linear interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0)$$

Since we want to find the velocity at $t = 16$, and we are using a first order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t = 15$ and $t = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$b_0 = v(t_0)$$
$$= 362.78$$

$$
\begin{aligned}
b_1 &= \frac{v(t_1) - v(t_0)}{t_1 - t_0} \\
&= \frac{517.35 - 362.78}{20 - 15} \\
&= 30.914
\end{aligned}
$$

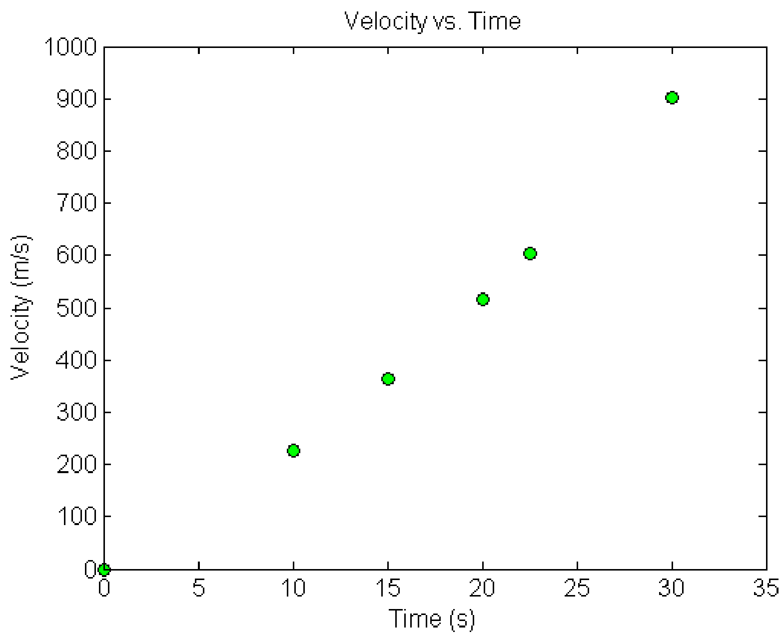


Figure 3 Graph of velocity vs. time data for the rocket example.

Hence

$$
\begin{aligned}
v(t) &= b_0 + b_1(t - t_0) \\
&= 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20
\end{aligned}
$$

At $t = 16$,

$$
\begin{aligned}
v(16) &= 362.78 + 30.914(16 - 15) \\
&= 393.69 \text{ m/s}
\end{aligned}
$$

If we expand

$$v(t) = 362.78 + 30.914(t - 15), \qquad 15 \le t \le 20$$

we get

$$v(t) = -100.93 + 30.914t, \qquad 15 \le t \le 20$$

and this is the same expression as obtained in the direct method.

Quadratic Interpolation

Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1)$$

$$= b_0$$

$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

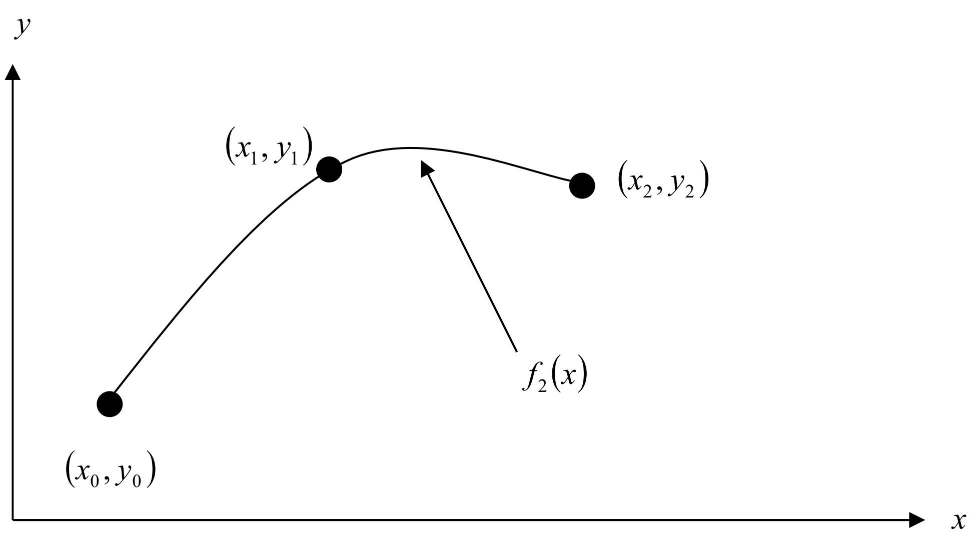


Figure 4 Quadratic interpolation.

Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using second order polynomial interpolation using Newton's divided difference polynomial method.

Solution

For quadratic interpolation, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

Since we want to find the velocity at $t = 16,$ and we are using a second order polynomial, we need to choose the three data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The three points are $t_0 = 10,$ $t_1 = 15,$ and $t_2 = 20$.

Then

$$t_0 = 10, \; v(t_0) = 227.04$$

$$t_1 = 15, \; v(t_1) = 362.78$$

$$t_2 = 20, \; v(t_2) = 517.35$$

gives

$$b_0 = v(t_0)$$

$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0}$$

$$= \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0}$$

$$= \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10}$$

$$= 0.37660$$

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$
$$= 392.19 \text{ m/s}$$

If we expand

$$v(t) = 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \ 10 \le t \le 20$$

we get

$$v(t) = 12.05 + 17.733t + 0.37660t^2, \quad 10 \le t \le 20$$

This is the same expression obtained by the direct method.

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that b_0, b_1, and b_2 are finite divided differences. b_0, b_1, and b_2 are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n + 1$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, as

$$f_n(x) = b_0 + b_1(x - x_0) + ... + b_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0]$$
$$b_n = f[x_n, x_{n-1}, ..., x_0]$$

where the definition of the m^{th} divided difference is

$$b_m = f[x_m, ..., x_0]$$
$$= \frac{f[x_m, ..., x_1] - f[x_{m-1}, ..., x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively. For an example of a third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3),

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

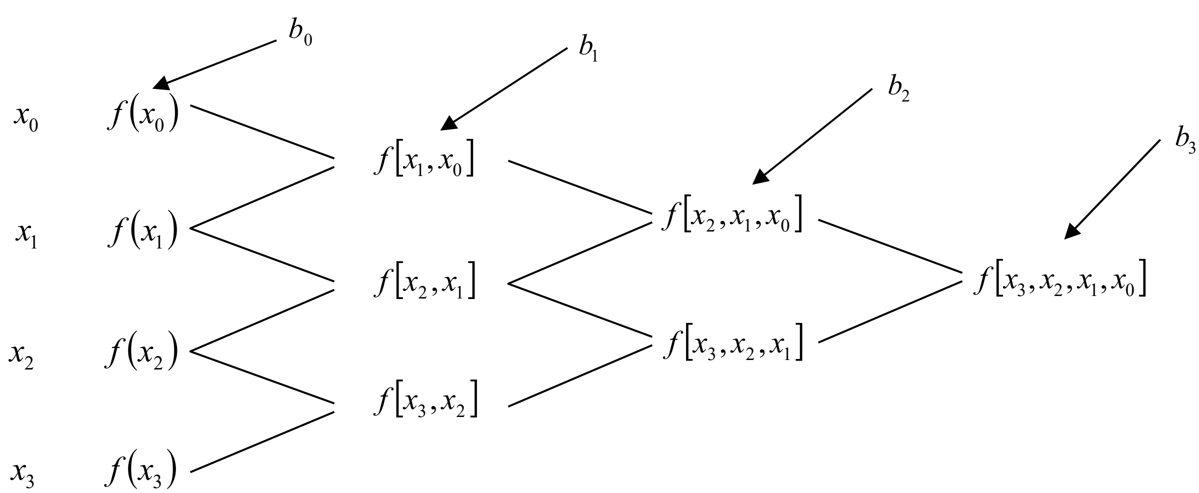


Figure 5 Table of divided differences for a cubic polynomial.

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78

20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds with third order polynomial interpolation using Newton's divided difference polynomial method.
b) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\,\text{s}$ to $t = 16\,\text{s}$.
c) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\,\text{s}$.

Solution
a) For a third order polynomial, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The four data points are $t_0 = 10,\ t_1 = 15,\ t_2 = 20,$ and $t_3 = 22.5$.

Then

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

gives

$$\begin{aligned} b_0 &= v[t_0] \\ &= v(t_0) \\ &= 227.04 \end{aligned}$$

$$\begin{aligned} b_1 &= v[t_1, t_0] \\ &= \frac{v(t_1) - v(t_0)}{t_1 - t_0} \\ &= \frac{362.78 - 227.04}{15 - 10} \\ &= 27.148 \end{aligned}$$

$$\begin{aligned} b_2 &= v[t_2, t_1, t_0] \\ &= \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0} \end{aligned}$$

$$\begin{aligned} v[t_2, t_1] &= \frac{v(t_2) - v(t_1)}{t_2 - t_1} \\ &= \frac{517.35 - 362.78}{20 - 15} \\ &= 30.914 \end{aligned}$$

$$v[t_1, t_0] = 27.148$$

$$b_2 = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$
$$= \frac{30.914 - 27.148}{20 - 10}$$
$$= 0.37660$$
$$b_3 = v[t_3, t_2, t_1, t_0]$$
$$= \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$
$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$
$$v[t_3, t_2] = \frac{v(t_3) - v(t_2)}{t_3 - t_2}$$
$$= \frac{602.97 - 517.35}{22.5 - 20}$$
$$= 34.248$$
$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$
$$= \frac{517.35 - 362.78}{20 - 15}$$
$$= 30.914$$
$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$
$$= \frac{34.248 - 30.914}{22.5 - 15}$$
$$= 0.44453$$
$$v[t_2, t_1, t_0] = 0.37660$$
$$b_3 = \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$
$$= \frac{0.44453 - 0.37660}{22.5 - 10}$$
$$= 5.4347 \times 10^{-3}$$

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$
$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15)$$
$$+ 5.5347 \times 10^{-3}(t - 10)(t - 15)(t - 20)$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$
$$+ 5.5347 \times 10^{-3}(16 - 10)(16 - 15)(16 - 20)$$
$$= 392.06 \text{ m/s}$$

b) The distance covered by the rocket between $t = 11\,\text{s}$ and $t = 16\,\text{s}$ can be calculated from the interpolating polynomial

$$\begin{aligned} v(t) &= 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15) \\ &\quad + 5.5347 \times 10^{-3}(t-10)(t-15)(t-20) \\ &= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3,\ 10 \le t \le 22.5 \end{aligned}$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of $t = 11$ and $t = 16$.

So

$$\begin{aligned} s(16) - s(11) &= \int_{11}^{16} v(t)dt \\ &= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)dt \\ &= \left[-4.2541t + 21.265\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16} \\ &= 1605\text{ m} \end{aligned}$$

c) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$

$$\begin{aligned} a(t) &= \frac{d}{dt}v(t) \\ &= \frac{d}{dt}\left(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3\right) \\ &= 21.265 + 0.26408t + 0.016304t^2 \end{aligned}$$

$$\begin{aligned} a(16) &= 21.265 + 0.26408(16) + 0.016304(16)^2 \\ &= 29.664\text{ m/s}^2 \end{aligned}$$

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Textbook notes on Newton's divided difference interpolation.
Major	General Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	Aralık 30, 2016
Web Site	http://numericalmethods.eng.usf.edu

Multiple-Choice Test Chapter 05.03 Newton's Divided Difference Polynomial Method

1. If a polynomial of degree n has $n+1$ zeros, then the polynomial is

(A) oscillatory
(B) zero everywhere
(C) quadratic

(D) not defined

2. The following x, y data is given.

x	15	18	22
y	24	37	25

The Newton's divided difference second order polynomial for the above data is given by

$$f_2(x) = b_0 + b_1(x-15) + b_2(x-15)(x-18)$$

The value of b_1 is most nearly

(A) -1.0480
(B) 0.14333
(C) 4.3333
(D) 24.000

3. The polynomial that passes through the following x, y data

x	18	22	24
y	?	25	123

is given by

$$8.125x^2 - 324.75x + 3237, \quad 18 \le x \le 24$$

The corresponding polynomial using Newton's divided difference polynomial is given by

$$f_2(x) = b_0 + b_1(x-18) + b_2(x-18)(x-22)$$

The value of b_2 is most nearly

(E) 0.25000
(F) 8.1250
(G) 24.000
(H) not obtainable with the information given

4. Velocity vs. time data for a body is approximated by a second order Newton's divided difference polynomial as

$$v(t) = b_0 + 39.622(t-20) + 0.5540(t-20)(t-15), \quad 10 \le t \le 20$$

The acceleration in m/s^2 at $t = 15$ is

(I) 0.5540
(J) 39.622
(K) 36.852
(L) not obtainable with the given information

5. The path that a robot is following on a $x-y$ plane is found by interpolating the following four data points as

x	2	4.5	5.5	7
y	7.5	7.5	6	5

$$y(x) = 0.1524x^3 - 2.257x^2 + 9.605x - 3.900$$

The length of the path from $x = 2$ to $x = 7$ is

(M) $\sqrt{(7.5-7.5)^2+(4.5-2)^2}+\sqrt{(6-7.5)^2+(5.5-4.5)^2}+\sqrt{(5-6)^2+(7-5.5)^2}$

(N) $\int_2^7 \sqrt{1+(0.1524x^3-2.257x^2+9.605x-3.900)^2}\,dx$

(O) $\int_2^7 \sqrt{1+(0.4572x^2-4.514x+9.605)^2}\,dx$

(P) $\int_2^7 (0.1524x^3-2.257x^2+9.605x-3.900)dx$

7. The following data of the velocity of a body is given as a function of time.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

If you were going to use quadratic interpolation to find the value of the velocity at $t = 14.9$ seconds, the three data points of time you would choose for interpolation are

(Q) 0, 15, 18
(R) 15, 18, 22
(S) 0, 15, 22
(T) 0, 18, 24

For a complete solution, refer to the links at the end of the book.

Newton's Divided Difference Interpolation – More Examples Chemical Engineering

Example 1

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at $61°\text{C}$. The specific heat of water is given as a function of time in Table 1.

Table 1 Specific heat of water as a function of temperature.

Temperature, T $(°\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg}-°\text{C}}\right)$

22	4181
42	4179
52	4186
82	4199
100	4217

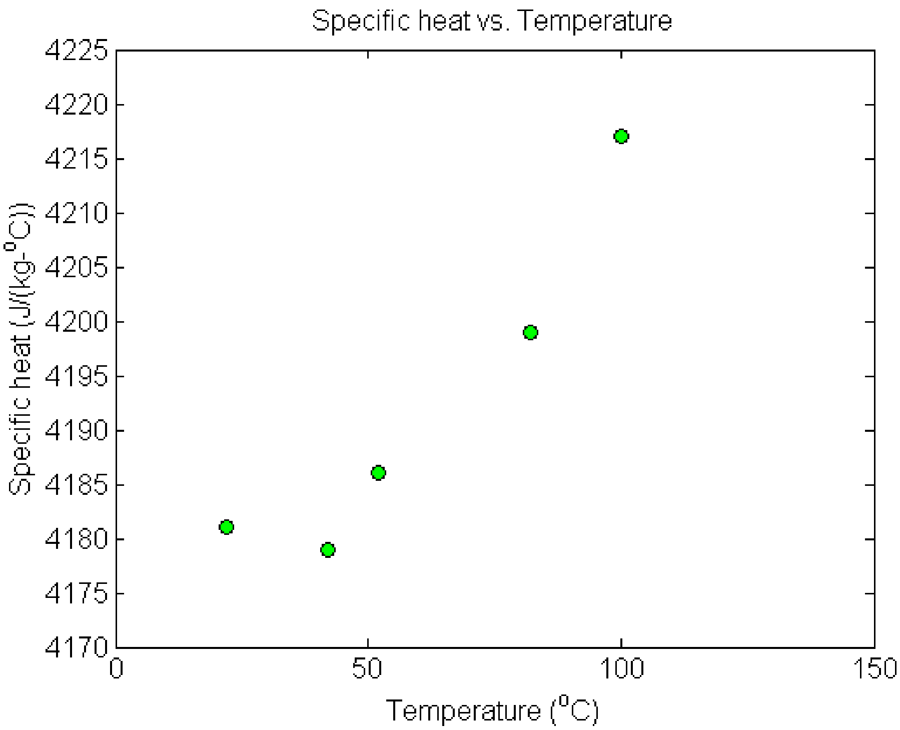


Figure 1 Specific heat of water vs. temperature.

Determine the value of the specific heat at $T = 61°\text{C}$ using Newton's divided difference method of interpolation and a first order polynomial.

Solution

For linear interpolation, the specific heat is given by

$$C_p(T) = b_0 + b_1(T - T_0)$$

Since we want to find the velocity at $T = 61°\text{C}$, and we are using a first order polynomial we need to choose the two data points that are closest to $T = 61°\text{C}$ that also bracket $T = 61°\text{C}$ to evaluate it. The two points are $T = 52$ and $T = 82$.

Then

$$T_0 = 52, \;\; C_p(T_0) = 4186$$

$$T_1 = 82, \;\; C_p(T_1) = 4199$$

gives

$$b_0 = C_p(T_0)$$

$$= 4186$$

$$b_1 = \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}$$
$$= \frac{4199 - 4186}{82 - 52}$$
$$= 0.43333$$

Hence

$$C_p(T) = b_0 + b_1(T - T_0)$$
$$= 4186 + 0.43333(T - 52), \qquad 52 \le T \le 82$$

At $T = 61$,

$$C_p(61) = 4186 + 0.43333(61 - 52)$$
$$= 4189.9 \frac{\text{J}}{\text{kg} - {}^\circ\text{C}}$$

If we expand

$$C_p(T) = 4186 + 0.43333(T - 52), \qquad 52 \le T \le 82$$

we get

$$C_p(T) = 4163.5 + 0.43333T, \qquad 52 \le T \le 82$$

and this is the same expression as obtained in the direct method.

Example 2

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61°C. The specific heat of water is given as a function of time in Table 2.

Table 2 Specific heat of water as a function of temperature.

Temperature, T $(^\circ\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} - {}^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at $T = 61^\circ\text{C}$ using Newton's divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For quadric interpolation, the specific heat is given by

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

Since we want to find the specific heat at $T = 61°\text{C}$, and we are using a second order polynomial, we need to choose the three data points that are closest to $T = 61°\text{C}$ that also bracket $T = 61°\text{C}$ to evaluate it. The three points are $T_0 = 42,\ T_1 = 52,$ and $T_2 = 82$.

Then

$$T_0 = 42,\ C_p(T_0) = 4179$$
$$T_1 = 52,\ C_p(T_1) = 4186$$
$$T_2 = 82,\ C_p(T_2) = 4199$$

gives

$$\begin{aligned} b_0 &= C_p(T_0) \\ &= 4179 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0} \\ &= \frac{4186 - 4179}{52 - 42} \\ &= 0.7 \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{\dfrac{C_p(T_2) - C_p(T_1)}{T_2 - T_1} - \dfrac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}}{T_2 - T_0} \\ &= \frac{\dfrac{4199 - 4186}{82 - 52} - \dfrac{4186 - 4179}{52 - 42}}{82 - 42} \\ &= \frac{0.43333 - 0.7}{40} \\ &= -6.6667 \times 10^{-3} \end{aligned}$$

Hence

$$\begin{aligned} C_p(T) &= b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) \\ &= 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52), \qquad 42 \le T \le 82 \end{aligned}$$

At $T = 61$,

$$\begin{aligned} C_p(61) &= 4179 + 0.7(61 - 42) - 6.6667 \times 10^{-3}(61 - 42)(61 - 52) \\ &= 4191.2 \frac{\text{J}}{\text{kg} - °\text{C}} \end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned} |\in_a| &= \left| \frac{4191.2 - 4189.9}{4191.2} \right| \times 100 \\ &= 0.030063\% \end{aligned}$$

If we expand

$$C_p(T) = 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52), \qquad 42 \le T \le 82$$

we get

$$C_p(T) = 4135.0 + 1.3267T - 6.6667 \times 10^{-3} T^2, \qquad 42 \le T \le 82$$

This is the same expression obtained by the direct method.

Example 3

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at $61°\text{C}$. The specific heat of water is given as a function of time in Table 3.

Table 3 Specific heat of water as a function of temperature.

Temperature, T $(°\text{C})$	Specific heat, C_p $\left(\frac{\text{J}}{\text{kg} - °\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at $T = 61\,°\text{C}$ using Newton's divided difference method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

Solution

For a third order polynomial, the specific heat profile is given by

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

Since we want to find the specific heat at $T = 61°\text{C}$, and we are using a third order polynomial, we need to choose the four data points that are closest to $T = 61°\text{C}$ that also bracket $T = 61°\text{C}$. The four data points are $T_0 = 42,\ T_1 = 52,\ T_2 = 82$ and $T_3 = 100$.

(Choosing the four points as $T_0 = 22$, $T_1 = 42$, $T_2 = 52$ and $T_3 = 82$ is equally valid.)

$$T_0 = 42, \quad C_p(T_0) = 4179$$

$$T_1 = 52, \quad C_p(T_1) = 4186$$

$$T_2 = 82, \quad C_p(T_2) = 4199$$

$$T_3 = 100, \quad C_p(T_3) = 4217$$

then

$$b_0 = C_p[T_0]$$

$$= C_p(T_0)$$

$$= 4179$$

$$b_1 = C_p[T_1, T_0]$$

$$= \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}$$
$$= \frac{4186 - 4179}{52 - 42}$$
$$= 0.7$$
$$b_2 = C_p[T_2, T_1, T_0]$$
$$= \frac{C_p[T_2, T_1] - C_p[T_1, T_0]}{T_2 - T_0}$$
$$C_p[T_2, T_1] = \frac{C_p(T_2) - C_p(T_1)}{T_2 - T_1}$$
$$= \frac{4199 - 4186}{82 - 52}$$
$$= 0.43333$$
$$C_p[T_1, T_0] = 0.7$$

$$b_2 = \frac{C_p[T_2, T_1] - C_p[T_1, T_0]}{T_2 - T_0}$$
$$= \frac{0.43333 - 0.7}{82 - 42}$$
$$= -6.6667 \times 10^{-3}$$

$$b_3 = C_p\left[T_3, T_2, T_1, T_0\right]$$
$$= \frac{C_p[T_3, T_2, T_1] - C_p[T_2, T_1, T_0]}{T_3 - T_0}$$
$$C_p[T_3, T_2, T_1] = \frac{C_p[T_3, T_2] - C_p[T_2, T_1]}{T_3 - T_1}$$
$$C_p[T_3, T_2] = \frac{C_p(T_3) - C_p(T_2)}{T_3 - T_2}$$
$$= \frac{4217 - 4199}{100 - 82}$$
$$= 1$$
$$C_p[T_2, T_1] = \frac{C_p(T_2) - C_p(T_1)}{T_2 - T_1}$$
$$= \frac{4199 - 4186}{82 - 52}$$
$$= 0.43333$$
$$C_p[T_3, T_2, T_1] = \frac{C_p[T_3, T_2] - C_p[T_2, T_1]}{T_3 - T_1}$$

$$= \frac{1 - 0.43333}{100 - 52}$$

$$= 0.011806$$

$$C_p[T_2, T_1, T_0] = -6.6667 \times 10^{-3}$$

$$b_3 = C_p[T_3, T_2, T_1, T_0]$$

$$= \frac{C_p[T_3, T_2, T_1] - C_p[T_2, T_1, T_0]}{T_3 - T_0}$$

$$= \frac{0.011806 + 6.6667 \times 10^{-3}}{100 - 42}$$

$$= 3.1849 \times 10^{-4}$$

Hence

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

$$= 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52)$$

$$+ 3.1849 \times 10^{-4}(T - 42)(T - 52)(T - 82), \qquad 42 \le T \le 100$$

At $T = 61$,

$$C_p(61) = 4179 + 0.7(61 - 42) - 6.6667 \times 10^{-3}(61 - 42)(61 - 52)$$

$$+ 3.1849 \times 10^{-4}(61 - 42)(61 - 52)(61 - 82)$$

$$= 4190.0 \frac{\text{J}}{\text{kg} - {}^\circ\text{C}}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left| \frac{4190.0 - 4191.2}{4190.0} \right| \times 100$$

$$= 0.027295\%$$

If we expand

$$C_p(T) = 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52)$$

$$+ 3.1849 \times 10^{-4}(T - 42)(T - 52)(T - 82), \qquad 42 \le T \le 100$$

we get

$$C_p(T) = 4078.0 + 4.4771T - 0.06272T^2 + 3.1849 \times 10^{-4} T^3, \; 42 \le T \le 100$$

This is the same expression as obtained in the direct method.

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Examples of Newton's divided difference interpolation.
Major	Chemical Engineering
Authors	Autar Kaw
Date	Aralık 30, 2016

1.6 Chapter 05.04 Lagrangian Interpolation

After reading this chapter, you should be able to:

1. *derive Lagrangian method of interpolation,*
2. *solve problems using Lagrangian method of interpolation, and*
3. *use Lagrangian interpolants to find derivatives and integrals of discrete functions.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(D) evaluate,
(E) differentiate, and
(F) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method. We discuss the Lagrangian method in this chapter.

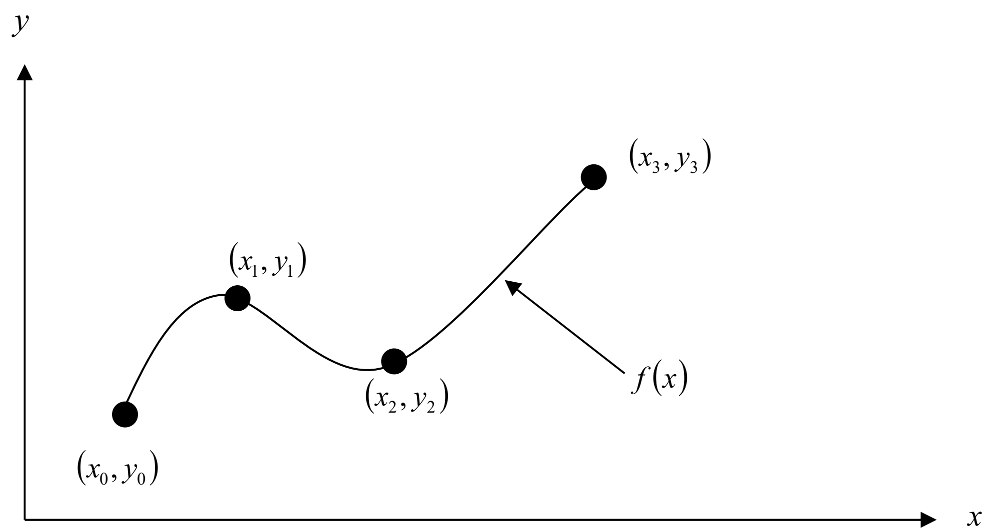


Figure 1 Interpolation of discrete data.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1
The upward velocity of a rocket is given as a function of time in Table 1.

Table 1 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

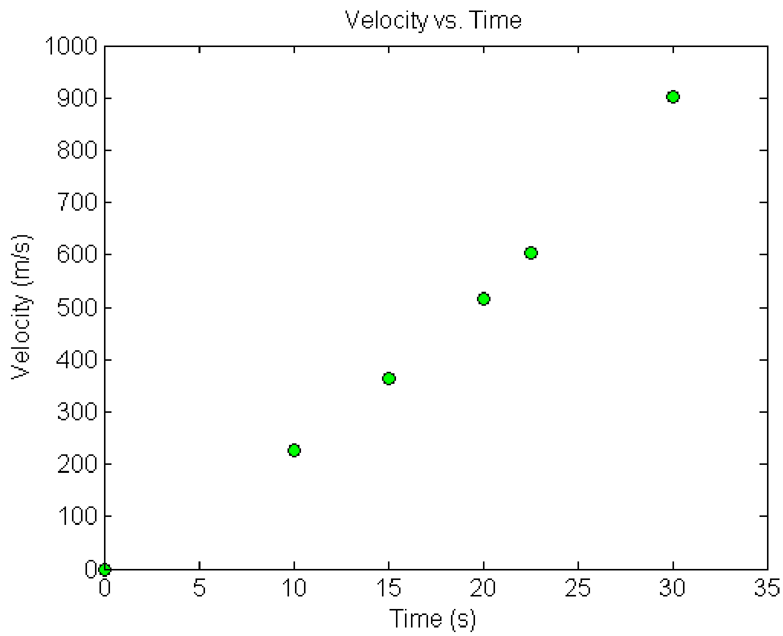


Figure 2 Graph of velocity vs. time data for the rocket example.

Determine the value of the velocity at $t = 16$ seconds using a first order Lagrange polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), the velocity is given by

$$\begin{aligned} v(t) &= \sum_{i=0}^{1} L_i(t)v(t_i) \\ &= L_0(t)v(t_0) + L_1(t)v(t_1) \end{aligned}$$

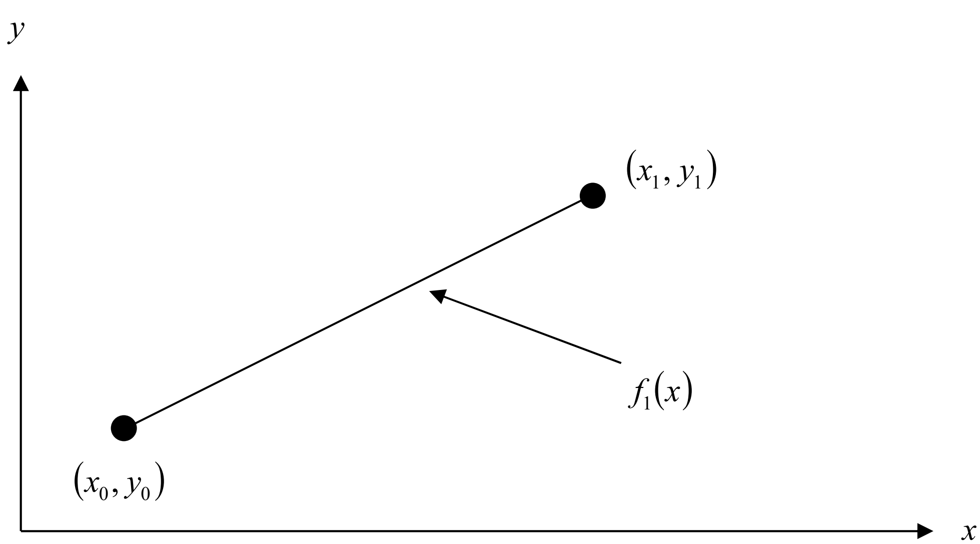


Figure 3 Linear interpolation.

Since we want to find the velocity at $t = 16$, and we are using a first order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78$$
$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{t - t_j}{t_0 - t_j}$$
$$= \frac{t - t_1}{t_0 - t_1}$$
$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{t - t_j}{t_1 - t_j}$$
$$= \frac{t - t_0}{t_1 - t_0}$$

Hence

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1)$$
$$= \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35), \quad 15 \le t \le 20$$
$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35)$$
$$= 0.8(362.78) + 0.2(517.35)$$
$$= 393.69 \text{ m/s}$$

You can see that $L_0(t) = 0.8$ and $L_1(t) = 0.2$ are like weightages given to the velocities at $t = 15$ and $t = 20$ to calculate the velocity at $t = 16$.

Quadratic Interpolation

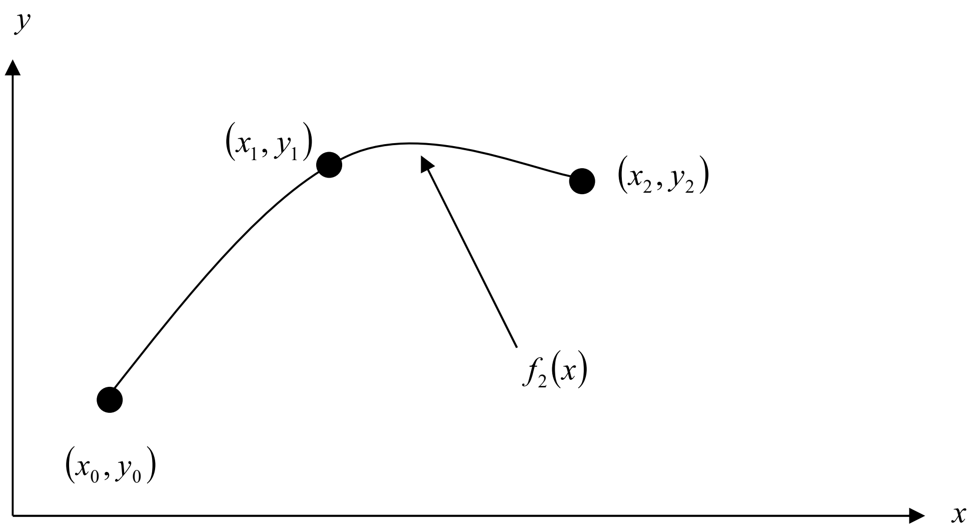


Figure 4 Quadratic interpolation.

Example 2
The upward velocity of a rocket is given as a function of time in Table 2.

Table 2 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds with second order polynomial interpolation using Lagrangian polynomial interpolation.
b) Find the absolute relative approximate error for the second order polynomial approximation.

Solution
a) For second order polynomial interpolation (also called quadratic interpolation), the velocity is given by

$$\begin{aligned} v(t) &= \sum_{i=0}^{2} L_i(t)v(t_i) \\ &= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) \end{aligned}$$

Since we want to find the velocity at $t = 16$, and we are using a second order polynomial, we need to choose the three data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The three points are $t_0 = 10$, $t_1 = 15$, and $t_2 = 20$.

Then

$$t_0 = 10, \; v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$
$$t_2 = 20,\ v(t_2) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t - t_j}{t_0 - t_j}$$
$$= \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)$$
$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j}$$
$$= \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$
$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j}$$
$$= \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)$$

Hence

$$v(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)v(t_0) + \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)v(t_1) + \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)v(t_2),\ t_0 \le t \le t_2$$
$$v(16) = \frac{(16-15)(16-20)}{(10-15)(10-20)}(227.04) + \frac{(16-10)(16-20)}{(15-10)(15-20)}(362.78)$$
$$+ \frac{(16-10)(16-15)}{(20-10)(20-15)}(517.35)$$
$$= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(517.35)$$
$$= 392.19 \text{ m/s}$$

b) The absolute relative approximate error $|\in_a|$ for the second order polynomial is calculated by considering the result of the first order polynomial (Example 1) as the previous approximation.

$$|\in_a| = \left|\frac{392.19 - 393.69}{392.19}\right| \times 100$$
$$= 0.38410\%$$

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04

15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds using third order Lagrangian polynomial interpolation.
b) Find the absolute relative approximate error for the third order polynomial approximation.
c) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\text{ s}$ to $t = 16\text{ s}$.
d) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\text{ s}$.

Solution
a) For third order polynomial interpolation (also called cubic interpolation), the velocity is given by

$$
\begin{aligned}
v(t) &= \sum_{i=0}^{3} L_i(t)v(t_i) \\
&= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)
\end{aligned}
$$

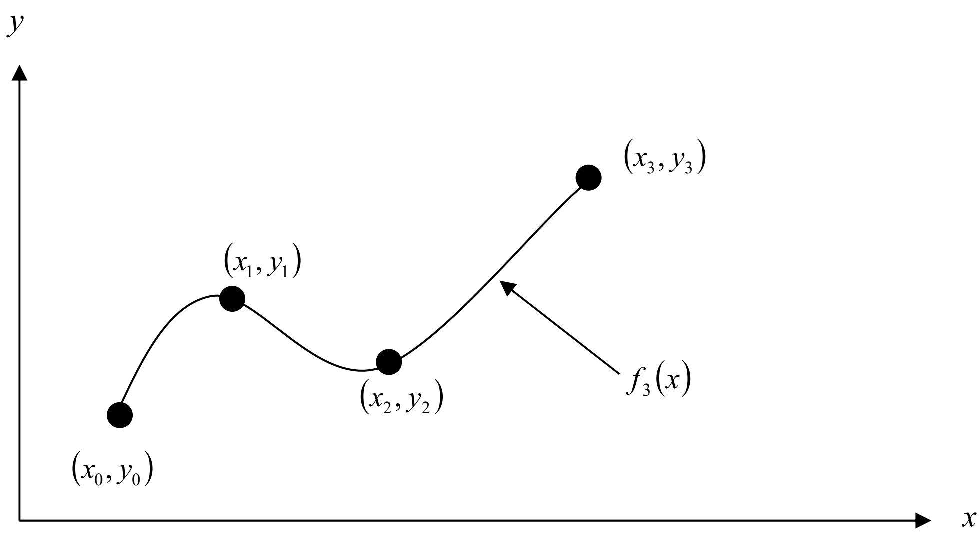


Figure 5 Cubic interpolation.

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four data points closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The four points are $t_0 = 10,\ t_1 = 15,\ t_2 = 20$ and $t_3 = 22.5$.
Then

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$
$$t_2 = 20,\ v(t_2) = 517.35$$
$$t_3 = 22.5,\ v(t_3) = 602.97$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{3} \frac{t - t_j}{t_0 - t_j}$$
$$= \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)\left(\frac{t - t_3}{t_0 - t_3}\right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{3} \frac{t - t_j}{t_1 - t_j}$$
$$= \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)\left(\frac{t - t_3}{t_1 - t_3}\right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^{3} \frac{t - t_j}{t_2 - t_j}$$
$$= \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)\left(\frac{t - t_3}{t_2 - t_3}\right)$$

$$L_3(t) = \prod_{\substack{j=0 \\ j \neq 3}}^{3} \frac{t - t_j}{t_3 - t_j}$$
$$= \left(\frac{t - t_0}{t_3 - t_0}\right)\left(\frac{t - t_1}{t_3 - t_1}\right)\left(\frac{t - t_2}{t_3 - t_2}\right)$$

Hence

$$v(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)\left(\frac{t - t_3}{t_0 - t_3}\right)v(t_0) + \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)\left(\frac{t - t_3}{t_1 - t_3}\right)v(t_1)$$
$$+ \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)\left(\frac{t - t_3}{t_2 - t_3}\right)v(t_2) + \left(\frac{t - t_0}{t_3 - t_0}\right)\left(\frac{t - t_1}{t_3 - t_1}\right)\left(\frac{t - t_2}{t_3 - t_2}\right)v(t_3),\ t_0 \le t \le t_3$$

$$v(16) = \frac{(16-15)(16-20)(16-22.5)}{(10-15)(10-20)(10-22.5)}(227.04) + \frac{(16-10)(16-20)(16-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+ \frac{(16-10)(16-15)(16-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+ \frac{(16-10)(16-15)(16-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97)$$
$$= (-0.0416)(227.04) + (0.832)(362.78) + (0.312)(517.35) + (-0.1024)(602.97)$$
$$= 392.06 \text{ m/s}$$

b) The absolute percentage relative approximate error, $|\in_a|$ for the value obtained for $v(16)$ can be obtained by comparing the result with that obtained using the second order polynomial (Example 2)

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$
$$= 0.033269\%$$

c) The distance covered by the rocket between $t = 11\,\text{s}$ to $t = 16\,\text{s}$ can be calculated from the interpolating polynomial as

$$v(t) = \frac{(t-15)(t-20)(t-22.5)}{(10-15)(10-20)(10-22.5)}(227.04) + \frac{(t-10)(t-20)(t-22.5)}{(15-10)(15-20)(15-22.5)}(362.78)$$
$$+ \frac{(t-10)(t-15)(t-22.5)}{(20-10)(20-15)(20-22.5)}(517.35)$$
$$+ \frac{(t-10)(t-15)(t-20)}{(22.5-10)(22.5-15)(22.5-20)}(602.97), 10 \le t \le 22.5$$
$$= \frac{(t^2 - 35t + 300)(t - 22.5)}{(-5)(-10)(-12.5)}(227.04) + \frac{(t^2 - 30t + 200)(t - 22.5)}{(5)(-5)(-7.5)}(362.78)$$
$$+ \frac{(t^2 - 25t + 150)(t - 22.5)}{(10)(5)(-2.5)}(517.35) + \frac{(t^2 - 25t + 150)(t - 20)}{(12.5)(7.5)(2.5)}(602.97)$$

$$= (t^3 - 57.5t^2 + 1087.5t - 6750)(-0.36326) + (t^3 - 52.5t^2 + 875t - 4500)(1.9348)$$
$$+ (t^3 - 47.5t^2 + 712.5t - 3375)(-4.1388) + (t^3 - 45t^2 + 650t - 3000)(2.5727)$$
$$= -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3,\ 10 \le t \le 22.5$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of $t = 11$ and $t = 16$.

So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$
$$= \int_{11}^{16} (-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3)dt$$
$$= \left[-4.245t + 21.265\frac{t^2}{2} + 0.13195\frac{t^3}{3} + 0.00544\frac{t^4}{4}\right]_{11}^{16}$$
$$= 1605\text{ m}$$

d) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$

Given that

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3,\ 10 \le t \le 22.5$$
$$a(t) = \frac{d}{dt}v(t)$$

$$= \frac{d}{dt}\left(-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3\right)$$
$$= 21.265 + 0.26390t + 0.01632t^2,\ 10 \le t \le 22.5$$
$$a(16) = 21.265 + 0.26390(16) + 0.01632(16)^2$$
$$= 29.665\ \text{m/s}^2$$

Note: There is no need to get the simplified third order polynomial expression to conduct the differentiation. An expression of the form

$$L_0(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)$$

gives the derivative without expansion as

$$\frac{d}{dt}\left(L_0(t)\right) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right) + \left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right) + \left(\frac{t-t_3}{t_0-t_3}\right)\left(\frac{t-t_1}{t_0-t_1}\right)$$

INTERPOLATION	
Topic	Lagrange Interpolation
Summary	Textbook notes on the Lagrangian method of interpolation
Major	General Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	Aralık 30, 2016
Web Site	http://numericalmethods.eng.usf.edu

Multiple-Choice Test Chapter 05.04 Lagrange Method of Interpolation

1. A unique polynomial of degree _______________ passes through $n+1$ data points.
   (A) $n+1$ (B) n (C) n or less (D) $n+1$ or less

2. Given the two points $[a, f(a)]$, $[b, f(b)]$, the linear Lagrange polynomial $f_1(x)$ that passes through these two points is given by

(A) $f_1(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{a-b}f(b)$ (B) $f_1(x) = \frac{x}{b-a}f(a) + \frac{x}{b-a}f(b)$

(C) $f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(b-a)$ (D) $f_1(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b)$

3. The Lagrange polynomial that passes through the 3 data points is given by

x	15	18	22
y	24	37	25

$$f_2(x) = L_0(x)(24) + L_1(x)(37) + L_2(x)(25)$$

The value of $L_1(x)$ at $x = 16$ is most nearly

(A) –0.071430 (B) 0.50000 (C) 0.57143 (D) 4.3333

4. The following data of the velocity of a body is given as a function of time.

Time (s)	10	15	18	22	24
Velocity (m/s)	22	24	37	25	123

A quadratic Lagrange interpolant is found using three data points, $t = 15$, 18 and 22. From this information, at what of the times given in seconds is the velocity of the body 26 m/s during the time interval of $t = 15$ to $t = 22$ seconds.

(A) 20.173 (B) 21.858 (C) 21.667 (D) 22.020

5. The path that a robot is following on a x, y plane is found by interpolating four data points as

x	2	4.5	5.5	7
y	7.5	7.5	6	5

$$y(x) = 0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000$$

The length of the path from $x = 2$ to $x = 7$ is

(A) $\sqrt{(7.5-7.5)^2 + (4.5-2)^2} + \sqrt{(6-7.5)^2 + (5.5-4.5)^2} + \sqrt{(5-6)^2 + (7-5.5)^2}$

(B) $\int_2^7 \sqrt{1 + (0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000)^2}\,dx$

(C) $\int_2^7 \sqrt{1 + (0.45714x^2 - 4.5142x + 9.6048)^2}\,dx$

(D) $\int_2^7 (0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000)\,dx$

6. The following data of the velocity of a body is given as a function of time.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

If you were going to use quadratic interpolation to find the value of the velocity at $t = 14.9$ seconds, what three data points of time would you choose for interpolation?

(A) 0, 15, 18 (B) 15, 18, 22 (C) 0, 15, 22 (D) 0, 18, 24

<u>For a complete solution, refer to the links at the end of the book.</u>

1.7 Chapter 05.05 Spline Method of Interpolation

After reading this chapter, you should be able to:

1. *interpolate data using spline interpolation, and*
2. *understand why spline interpolation is important.*

What is interpolation?

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*. (çoğu zaman veriler (x_0, y_0), (x_1, y_1), …., (x_{n-1}, y_{n-1}), (x_n, y_n) kesikli noktalar şeklindedir. Bu durumda herhangi bir x değerine karşılık gelecek olan y nasıl elde edilir. F(x) şeklindeki sürekli bir fonksiyonu n+1 noktanın hepsinden geçirilebilir (Şekil 1). Böylelikle herhangi bir x değerine karşılık gelen y değeri bulunabilir. Buna interpolasyo demiştik.)

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*. x değeri sınırlar dışında kalıyorsa interpolasyon yerine extraolasyon ifadesini kullanabiliriz.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to (f(x) nasıl olmalıdır? Polinom interpolasyon fonksiyonu f(x) için polinomlar trigonometrik ve üstel serilere göre daha kullanışlıdır:)

(A) evaluate, (geliştirilebilir)
(B) differentiate, and (türevlenebilir)
(C) integrate (toplanabilir/integre edilebilir)

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. Several methods to obtain such a polynomial include the direct method, Newton's divided difference polynomial method and the Lagrangian interpolation method. (polinomiyal interpolasyon n. dereceden ve n+1 veri noktasından geçen polinomdur. Bu tür polinomları elde edebilmek için doğrudan, Newton bölümlü fark veya Lagrangian polinom yöntemleri kullanılabilir.)

So is the spline method yet another method of obtaining this n^{th} order polynomial. …… NO! Actually, when n becomes large, in many cases, one may get oscillatory behavior in the resulting polynomial. This was shown by Runge when he interpolated data based on a simple function of (Spline yöntemi n.dereceden polinom elde edebilmek için kullanılabilecek bir yöntemdir. Burada dikkat edilmesi gereken şey n derecesi büyüdükçe polinomdan elde edilen değerlerin osilasyon yapmaktadır. Bu osilasyon Runge tarafından aşağıdaki fonksiyondan elde edilen değerler kullanılarak gösterilmiştir. Bu fonksiyondan [-1, 1] aralığında elde edilmiş 6 değeri kullanılır ve bu değerlere uygun interpolasyon polinomu elde edilmeya çalışılırsa.)

$$y = \frac{1}{1+25x^2}$$

on an interval of [–1, 1]. For example, take six equidistantly spaced points in [–1, 1] and find y at these points as given in Table 1.

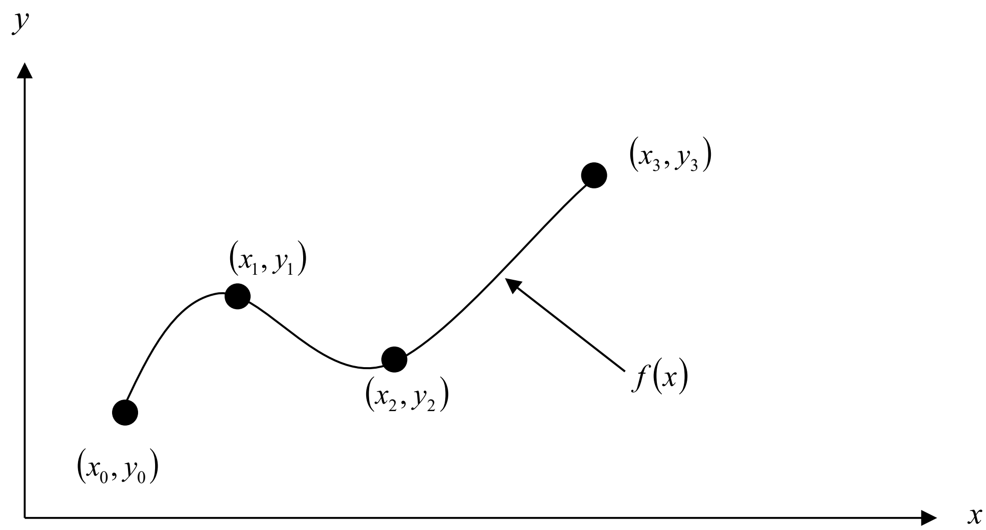


Figure 1 Interpolation of discrete data.

Table 1 Six equidistantly spaced points in [–1, 1].

x	$y = \frac{1}{1+25x^2}$
–1.0	0.038461
–0.6	0.1
–0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

Now through these six points, one can pass a fifth order polynomial

$$f_5(x) = 3.1378\times10^{-11}x^5 + 1.2019x^4 - 3.3651\times10^{-11}x^3 - 1.7308x^2 + 1.0004\times10^{-11}x + 5.6731\times10^{-1},$$
$$-1 \le x \le 1$$

through the six data points. On plotting the fifth order polynomial (Figure 2) and the original function, one can see that the two do not match well. One may consider choosing more points in the interval [–1, 1] to get a better match, but it diverges even more (see Figure 3), where 20 equidistant points were chosen in the interval [–1, 1] to draw a 19th order polynomial. In fact, Runge found that as the order of the polynomial becomes infinite, the polynomial diverges in the interval of $-1 < x < -0.726$ and $0.726 < x < 1$. (5.dereceden bir polinom yukarıdaki veri noktalarından geçmektedir. Bu 5. dereceden polinom kullanılarak veri noktalarından geçen bir grafik çizilecek olursa Şekil 2 deki gibi bir grafik elde edilir. Şekilden görüleceği gibi polinom verilere uygun değildir.)

So what is the answer to using information from more data points, but at the same time keeping the function true to the data behavior? The answer is in spline interpolation. The most common spline interpolations used are linear, quadratic, and cubic splines.

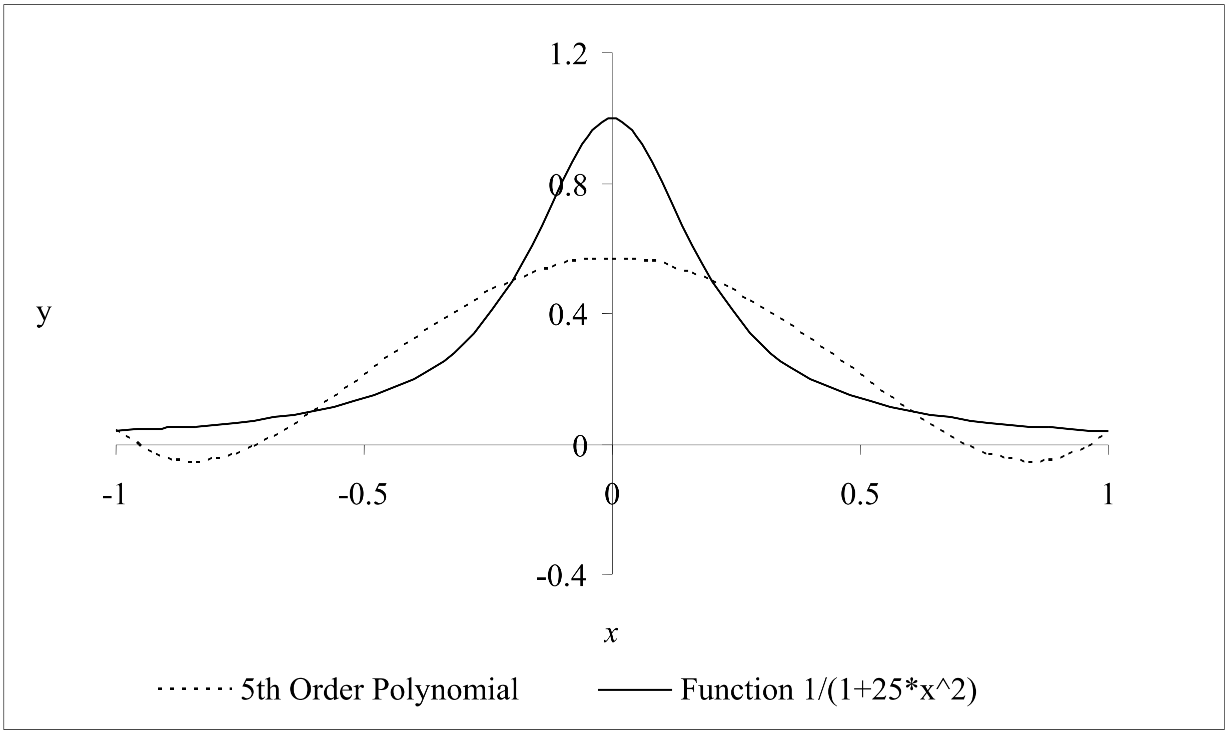


Figure 2 5th order polynomial interpolation with six equidistant points.

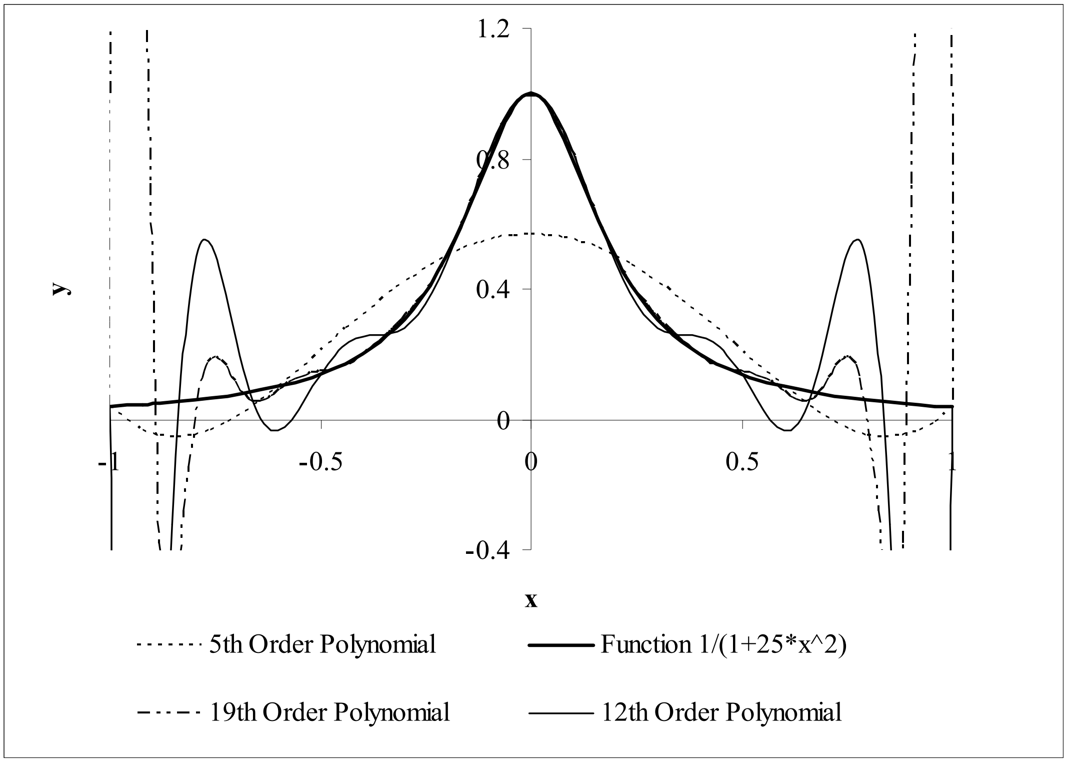


Figure 3 Higher order polynomial interpolation is a bad idea.

Linear Spline Interpolation (çizgisel spline interpolasyonu)

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines (Figure 4) to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $y_i = f(x_i)$. ((x_0,y_0), x_1,y_1), (x_2,y_2), ..., (x_{n-1},y_{n-1}), (x_n,y_n) veriler Şekil 4'deki gibi çizgisel spline'lara fit edilmektedir. Bu ardışık verilerin birbirleri ile çizgisel olarak birleştirilmesi işlemidir. Veriler artan sırada verilirse çizgisel spline y_i=f(x_i) şeklindedir.)

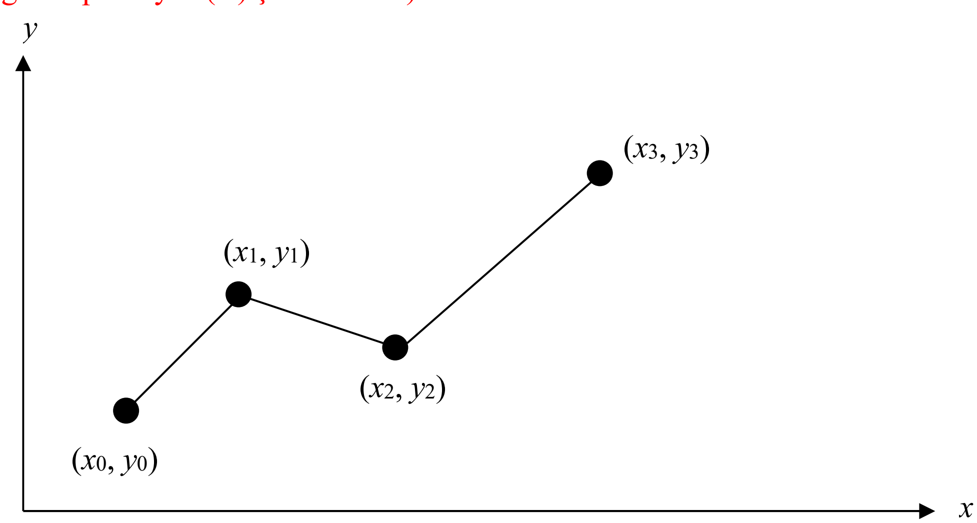


Figure 4 Linear splines.

$$
\begin{aligned}
f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1 \\
&= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n
\end{aligned}
$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example 1

The upward velocity of a rocket is given as a function of time in Table 2 (Figure 5).

Table 2 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

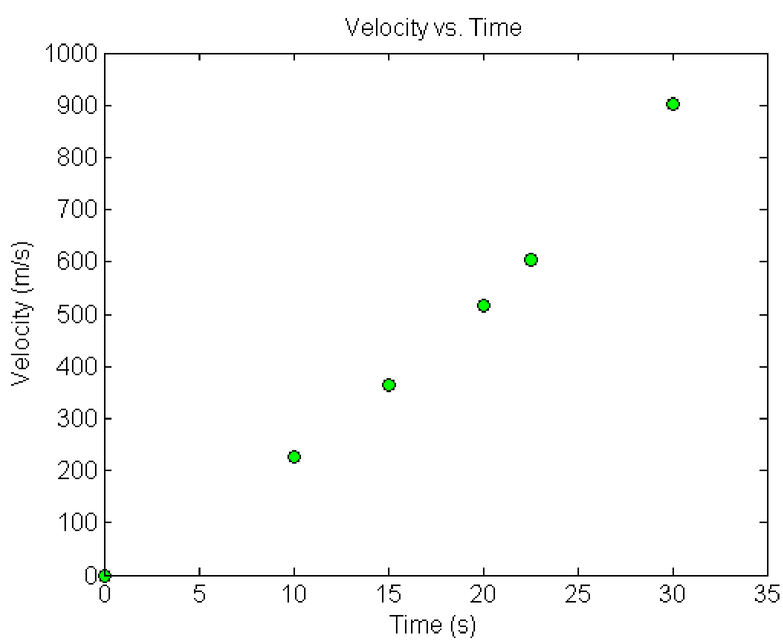


Figure 5 Graph of velocity vs. time data for the rocket example.

Determine the value of the velocity at $t = 16$ seconds using linear splines.

Solution

Since we want to evaluate the velocity at $t = 16$, and we are using linear splines, we need to choose the two data points closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78 \qquad t_1 = 20,\ v(t_1) = 517.35$$

gives

$$\begin{aligned} v(t) &= v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0) \\ &= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15) \\ &= 362.78 + 30.913(t - 15),\ 15 \le t \le 20 \end{aligned}$$

At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15) = 393.7\,\text{m/s}$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Spline kelimesi eğrilebilen, değişik şekillere sokulabilen çubuk veya dal anlamında kullanılmaktadır. Sayısal uygulamalarda bir veri grubunda veriler arasında yumuşak geçiş sağlayan fonksiyonlar olarak tanımlanabilir. N nci dereceden bir spline için N+1 tane veri noktasına ihtiyaç vardır. Burada N=1 için çizgisel, N=2 için kare ve N=3 için kübik spline fonksiyonları (2nci ve 3ncü dereceden polinomlar) ele alınacaktır.

Veriler için $v(t)=a_1+b_1t$ çizgisel fonksiyonu önerelim. 6 tane veri varsa 5 aralık/bölme vardır. her bölme için bu çizgisel denklemi yazarsak aşağıdaki çizelgeki değerler elde edilir. 6 veri varsa 12 bilinmeyen vardır. 12 tane denklem oluşturulursa bu veri grubu için birbiri ile uyumlu spline eğrileri elde edilir.

		i	t (s)	$v(t)$ (m/s)
		1	0	0.00
1.aralık	$v^{(1)}(t)=a_1^{(1)}+b_1^{(1)}t$			
		2	10	227.04
2.aralık	$v^{(2)}(t)=a_1^{(2)}+b_1^{(2)}t$			
		3	15	362.78
3.aralık	$v^{(3)}(t)=a_1^{(3)}+b_1^{(3)}t$			
		4	20	517.35
4.aralık	$v^{(4)}(t)=a_1^{(4)}+b_1^{(4)}t$			
		5	22.5	602.97
5.aralık	$v^{(5)}(t)=a_1^{(5)}+b_1^{(5)}t$			
		6	30	901.67

1.nokta için	$0 = a_1^{(0)}$ ilk nokta olduğu için
2.nokta için	$227.04 = a_1^{(1)} + b_1^{(1)} 10$ $227.04 = a_1^{(2)} + b_1^{(2)} 10$
3.nokta için	$362.78 = a_1^{(2)} + b_1^{(2)} 15$ $362.78 = a_1^{(3)} + b_1^{(3)} 15$
4.nokta için	$517.35 = a_1^{(3)} + b_1^{(3)} 20$ $517.35 = a_1^{(4)} + b_1^{(4)} 20$
5.nokta için	$602.97 = a_1^{(4)} + b_1^{(4)} 22.5$ $602.97 = a_1^{(5)} + b_1^{(5)} 22.5$
6.nokta için	$901.67 = a_1^{(6)} + b_1^{(6)} 30$ $b_1^{(6)} = 0$ alınır

Bu şekilde 10 bilinmeyenli 10 eşitliği olan bir denklem elde edilmiş olur.

Quadratic Splines

In these splines, a quadratic polynomial approximates the data between two consecutive data points. Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$\begin{aligned} f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\ &= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\ &\ . \\ &\ . \\ &\ . \\ &= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n \end{aligned}$$

So how does one find the coefficients of these quadratic splines? There are $3n$ such coefficients

$$a_i,\ i = 1,2,\ldots\ldots,n$$

$$b_i,\ i = 1,2,\ldots\ldots,n$$

$$c_i,\ i = 1,2,\ldots\ldots,n$$

To find $3n$ unknowns, one needs to set up $3n$ equations and then simultaneously solve them. These $3n$ equations are found as follows.

1. Each quadratic spline goes through two consecutive data points

$$a_1 x_0^{\ 2} + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^{\ 2} + b_1 x_1 + c_1 = f(x_1)$$

$$.$$
$$.$$
$$.$$

$$a_i x_{i-1}^{\ 2} + b_i x_{i-1} + c_i = f(x_{i-1})$$

$$a_i x_i^{\ 2} + b_i x_i + c_i = f(x_i)$$

$$.$$
$$.$$
$$.$$

$$a_n x_{n-1}^{\ 2} + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^{\ 2} + b_n x_n + c_n = f(x_n)$$

This condition gives $2n$ equations as there are n quadratic splines going through two consecutive data points.

2. The first derivatives of two quadratic splines are continuous at the interior points. For example, the derivative of the first spline

$$a_1 x^2 + b_1 x + c_1$$

is

$$2a_1 x + b_1$$

The derivative of the second spline

$$a_2 x^2 + b_2 x + c_2$$

is

$$2a_2 x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1 x_1 + b_1 = 2a_2 x_1 + b_2$$

$$2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$

Similarly at the other interior points,

$$2a_2 x_2 + b_2 - 2a_3 x_2 - b_3 = 0$$

$$\vdots$$

$$2a_i x_i + b_i - 2a_{i+1} x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1} x_{n-1} + b_{n-1} - 2a_n x_{n-1} - b_n = 0$$

Since there are $(n-1)$ interior points, we have $(n-1)$ such equations. So far, the total number of equations is $(2n) + (n-1) = (3n-1)$ equations. We still then need one more equation.

We can assume that the first spline is linear, that is

$$a_1 = 0$$

This gives us $3n$ equations and $3n$ unknowns. These can be solved by a number of techniques used to solve simultaneous linear equations.

Example 2

The upward velocity of a rocket is given as a function of time as

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

(a) Determine the value of the velocity at $t = 16$ seconds using quadratic splines.
(b) Using the quadratic splines as velocity functions, find the distance covered by the rocket from $t = 11\text{s}$ to $t = 16\text{s}$.
(c) Using the quadratic splines as velocity functions, find the acceleration of the rocket at $t = 16\text{s}$.

Solution
a) Since there are six data points, five quadratic splines pass through them.

$$\begin{aligned} v(t) &= a_1 t^2 + b_1 t + c_1, \quad 0 \le t \le 10 \\ &= a_2 t^2 + b_2 t + c_2, \quad 10 \le t \le 15 \\ &= a_3 t^2 + b_3 t + c_3, \quad 15 \le t \le 20 \\ &= a_4 t^2 + b_4 t + c_4, \quad 20 \le t \le 22.5 \\ &= a_5 t^2 + b_5 t + c_5, \quad 22.5 \le t \le 30 \end{aligned}$$

The equations are found as follows.
1. Each quadratic spline passes through two consecutive data points.
$a_1 t^2 + b_1 t + c_1$ passes through $t = 0$ and $t = 10$.

$$a_1 (0)^2 + b_1 (0) + c_1 = 0 \qquad (1)$$
$$a_1 (10)^2 + b_1 (10) + c_1 = 227.04 \qquad (2)$$

$a_2 t^2 + b_2 t + c_2$ passes through $t = 10$ and $t = 15$.

$$a_2 (10)^2 + b_2 (10) + c_2 = 227.04 \qquad (3)$$
$$a_2 (15)^2 + b_2 (15) + c_2 = 362.78 \qquad (4)$$

$a_3 t^2 + b_3 t + c_3$ passes through $t = 15$ and $t = 20$.

$$a_3 (15)^2 + b_3 (15) + c_3 = 362.78 \qquad (5)$$
$$a_3 (20)^2 + b_3 (20) + c_3 = 517.35 \qquad (6)$$

$a_4 t^2 + b_4 t + c_4$ passes through $t = 20$ and $t = 22.5$.

$$a_4 (20)^2 + b_4 (20) + c_4 = 517.35 \qquad (7)$$
$$a_4 (22.5)^2 + b_4 (22.5) + c_4 = 602.97 \qquad (8)$$

$a_5 t^2 + b_5 t + c_5$ passes through $t = 22.5$ and $t = 30$.

$$a_5 (22.5)^2 + b_5 (22.5) + c_5 = 602.97 \qquad (9)$$
$$a_5 (30)^2 + b_5 (30) + c_5 = 901.67 \qquad (10)$$

2. Quadratic splines have continuous derivatives at the interior data points.
At $t = 10$

$$2a_1 (10) + b_1 - 2a_2 (10) - b_2 = 0 \qquad (11)$$

At $t = 15$

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0 \quad (12)$$

At $t = 20$

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0 \quad (13)$$

At $t = 22.5$

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0 \quad (14)$$

3. Assuming the first spline $a_1t^2 + b_1t + c_1$ is linear,

$$a_1 = 0 \quad (15)$$

Combining Equation (1) –(15) in matrix form gives

$$\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 15 equations give the 15 unknowns as

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	–0.1356	35.66	–141.61
4	1.6048	–33.956	554.55
5	0.20889	28.86	–152.13

Therefore, the splines are given by

$$\begin{aligned}
v(t) &= 22.704t, && 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, && 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, && 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, && 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, && 22.5 \le t \le 30
\end{aligned}$$

At $t = 16\text{s}$

$$v(16) = -0.1356(16)^2 + 35.66(16) - 141.61 = 394.24\,\text{m/s}$$

b) The distance covered by the rocket between 11 and 16 seconds can be calculated as

$$s(16)-s(11)=\int_{11}^{16} v(t)dt$$

But since the splines are valid over different ranges, we need to break the integral accordingly as

$$v(t)=0.8888t^2+4.928t+88.88, \quad 10 \le t \le 15$$
$$=-0.1356t^2+35.66t-141.61, \quad 15 \le t \le 20$$

$$\int_{11}^{16} v(t)dt=\int_{11}^{15} v(t)dt+\int_{15}^{16} v(t)dt$$

$$s(16)-s(11)=\int_{11}^{15}(0.8888t^2+4.928t+88.88)dt+\int_{15}^{16}(-0.1356t^2+35.66t-141.61)dt$$

$$=\left[0.8888\frac{t^3}{3}+4.928\frac{t^2}{2}+88.88t\right]_{11}^{15}$$
$$+\left[-0.1356\frac{t^3}{3}+35.66\frac{t^2}{2}-141.61t\right]_{15}^{16}$$
$$=1217.35+378.53=1595.9 \text{ m}$$

c) What is the acceleration at $t=16$?

$$a(16)=\frac{d}{dt}v(t)\Big|_{t=16}$$
$$a(t)=\frac{d}{dt}v(t)=\frac{d}{dt}(-0.1356t^2+35.66t-141.61)$$
$$=-0.2712t+35.66, \qquad 15 \le t \le 20$$
$$a(16)=-0.2712(16)+35.66=31.321\text{m/s}^2$$

INTERPOLATION	
Topic	Spline Method of Interpolation
Summary	Textbook notes on the spline method of interpolation
Major	General Engineering
Authors	Autar Kaw, Michael Keteltas
Date	Aralık 30, 2016
Web Site	http://numericalmethods.eng.usf.edu

Multiple-Choice Test Chapter 05.05 Spline Method of Interpolation

1. The following n data points, (x_1,y_1), (x_2,y_2), …….. (x_n,y_n), are given. For conducting quadratic spline interpolation the x-data needs to be

(A) equally spaced
(B) placed in ascending or descending order of x-values
(C) integers
(D) positive

2. In cubic spline interpolation,

(A) the first derivatives of the splines are continuous at the interior data points
(B) the second derivatives of the splines are continuous at the interior data points
(C) the first and the second derivatives of the splines are continuous at the interior data points
(D) the third derivatives of the splines are continuous at the interior data points

3. The following incomplete y vs. x data is given.

x	1	2	4	6	7
y	5	11	????	????	32

The data is fit by quadratic spline interpolants given by

$f(x) = ax - 1, \quad 1 \le x \le 2$

$f(x) = -2x^2 + 14x - 9, \quad 2 \le x \le 4$

$f(x) = bx^2 + cx + d, \quad 4 \le x \le 6$

$f(x) = 25x^2 - 303x + 928, \quad 6 \le x \le 7$

where $a, b, c,$ and d are constants. The value of c is most nearly

(A) -303.00 (B) -144.50 (C) 0.0000 (D) 14.000

4. The following incomplete y vs. x data is given.

x	1	2	4	6	7
y	5	11	????	????	32

The data is fit by quadratic spline interpolants given by

$f(x) = ax - 1, \quad 1 \le x \le 2,$

$f(x) = -2x^2 + 14x - 9, \quad 2 \le x \le 4$

$f(x) = bx^2 + cx + d, \quad 4 \le x \le 6$

$f(x) = ex^2 + fx + g, \quad 6 \le x \le 7$

where $a, b, c, d, e, f,$ and g are constants. The value of $\frac{df}{dx}$ at $x = 2.6$ most nearly is

(A) -144.50 (B) -4.0000 (C) 3.6000 (D) 12.200

5. The following incomplete y vs. x data is given.

x	1	2	4	6	7
y	5	11	????	????	32

The data is fit by quadratic spline interpolants given by

$f(x) = ax - 1, \quad 1 \le x \le 2,$

$f(x) = -2x^2 + 14x - 9, \quad 2 \le x \le 4$

$f(x) = bx^2 + cx + d, \quad 4 \le x \le 6$

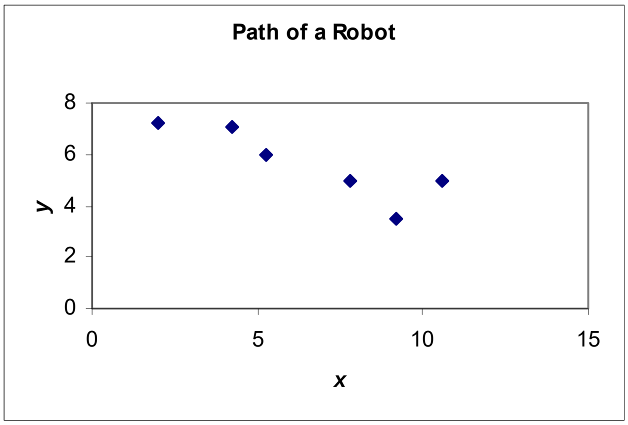
$f(x) = 25x^2 - 303x + 928, \quad 6 \le x \le 7$

where $a, b, c,$ and d are constants. What is the value of $\int_{1.5}^{3.5} f(x)dx$?

(A) 23.500 (B) 25.667 (C) 25.750 (D) 28.000

6. A robot needs to follow a path that passes consecutively through six points as shown in the figure. To find the shortest path that is also smooth you would recommend which of the following? (bir robot ardışık 6 noktadan geçecektir. En kısa yolu bulabilmek için aşağıdakilerden hangisini tavsiye edersiniz?)

(A) Pass a fifth order polynomial through the data
(B) Pass linear splines through the data
(C) Pass quadratic splines through the data
(D) Regress the data to a second order polynomial



For a complete solution, refer to the links at the end of the book.

Chapter 08.02 Euler's Method for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. *develop Euler's Method for solving ordinary differential equations,*
2. *determine how the step size affects the accuracy of a solution,*
3. *derive Euler's formula from Taylor series, and*
4. *use Euler's method to find approximate values of integrals.*

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \tag{1}$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ y(0) = y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx}+x^2y^2=2\sin(3x),\ y(0)=5$$

$$\frac{dy}{dx}=\frac{2\sin(3x)-x^2y^2}{e^y},\ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^2y^2}{e^y}$$

Derivation of Euler's method

At $x=0$, we are given the value of $y=y_0$. Let us call $x=0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x=x_0$, the slope is $f(x_0,y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0)=y_0$.

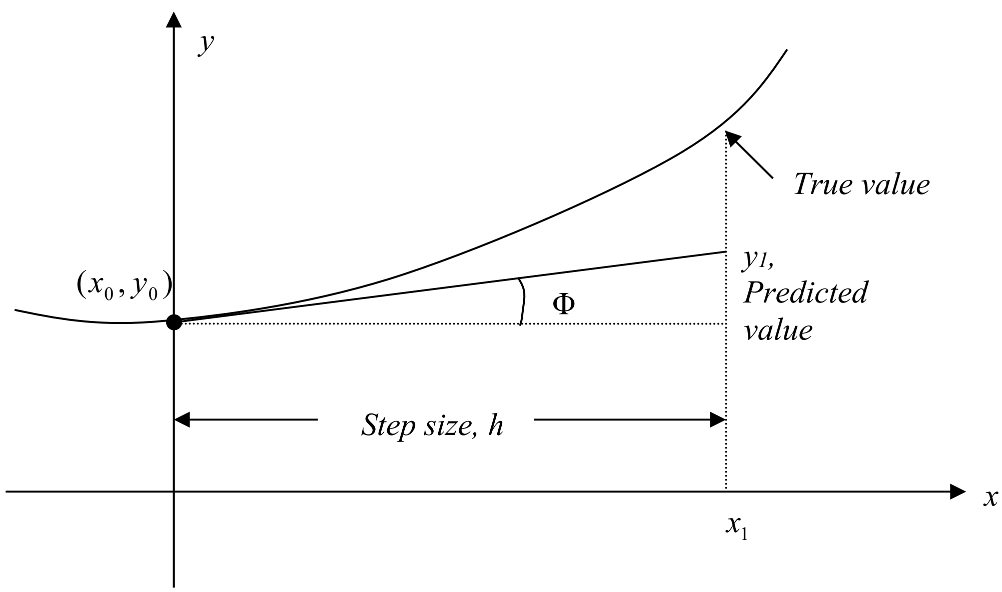


Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x=x_0$ as shown in Figure 1 is

$$\begin{aligned}\text{Slope} &= \frac{Rise}{Run}\\ &= \frac{y_1-y_0}{x_1-x_0}\\ &= f(x_0,y_0)\end{aligned}$$

From here

$$y_1=y_0+f(x_0,y_0)(x_1-x_0)$$

Calling x_1-x_0 the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h \qquad (2)$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)h$$

$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad (3)$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

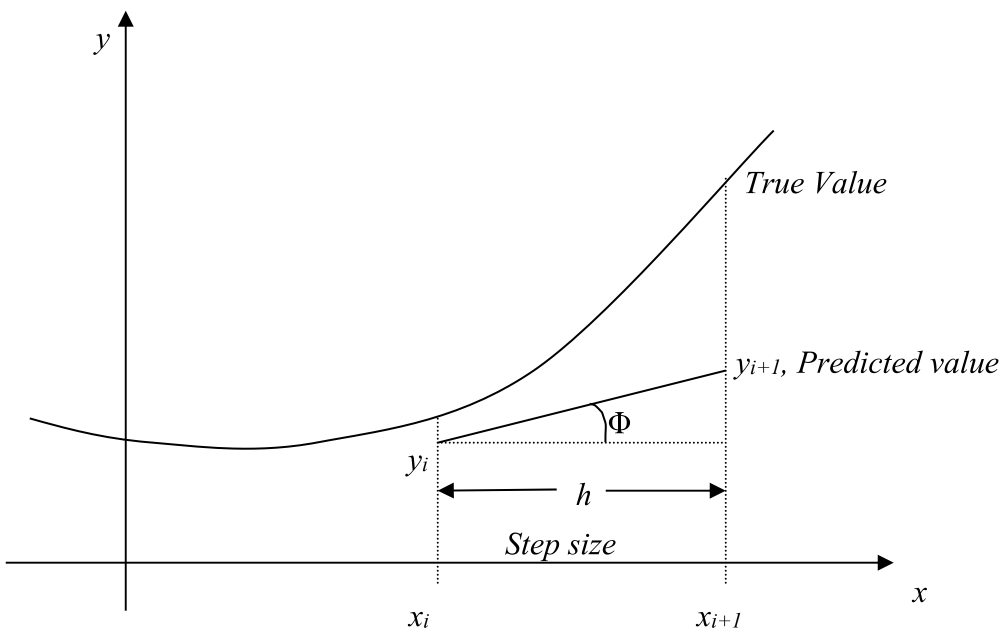


Figure 2 General graphical interpretation of Euler's method.

Example 3

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right), \ \theta(0) = 1200\text{K}$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$f(t, \theta) = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

Per Equation (3), Euler's method reduces to

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200$

$$\theta_1 = \theta_0 + f(t_0, \theta_0)h$$
$$= 1200 + f(0,1200) \times 240$$
$$= 1200 + \left(-2.2067 \times 10^{-12}\left(1200^4 - 81 \times 10^8\right)\right) \times 240$$
$$= 1200 + (-4.5579) \times 240$$
$$= 106.09\ \text{K}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$
$$\theta_1 = \theta(240) \approx 106.09\ \text{K}$$

For $i = 1$, $t_1 = 240$, $\theta_1 = 106.09$

$$\theta_2 = \theta_1 + f(t_1, \theta_1)h$$
$$= 106.09 + f(240,106.09) \times 240$$
$$= 106.09 + \left(-2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right)\right) \times 240$$
$$= 106.09 + (0.017595) \times 240$$
$$= 110.32\ \text{K}$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 240 + 240 = 480$$
$$\theta_2 = \theta(480) \approx 110.32\ \text{K}$$

Figure 3 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 240$.

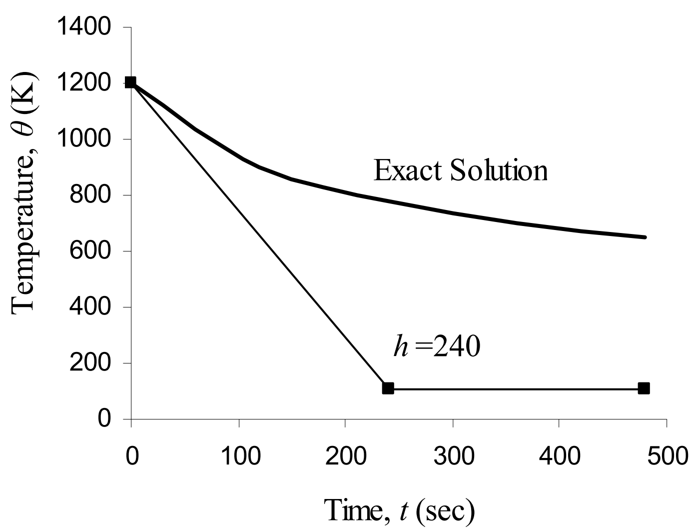


Figure 3 Comparing the exact solution and Euler's method.

The problem was solved again using a smaller step size. The results are given below in Table 1.

Table 1 Temperature at 480 seconds as a function of step size, h.

Step size, h	$\theta(480)$	E_t	$\lvert\in_t\rvert$ %
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

Figure 4 shows how the temperature varies as a function of time for different step sizes.

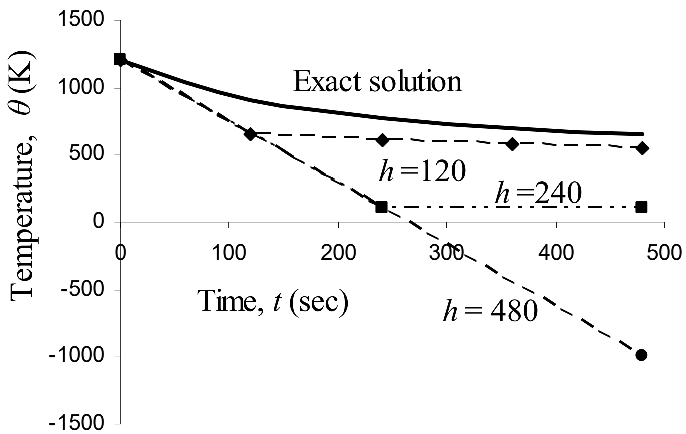


Figure 4 Comparison of Euler's method with the exact solution for different step sizes.

The values of the calculated temperature at $t = 480\,\text{s}$ as a function of step size are plotted in Figure 5.

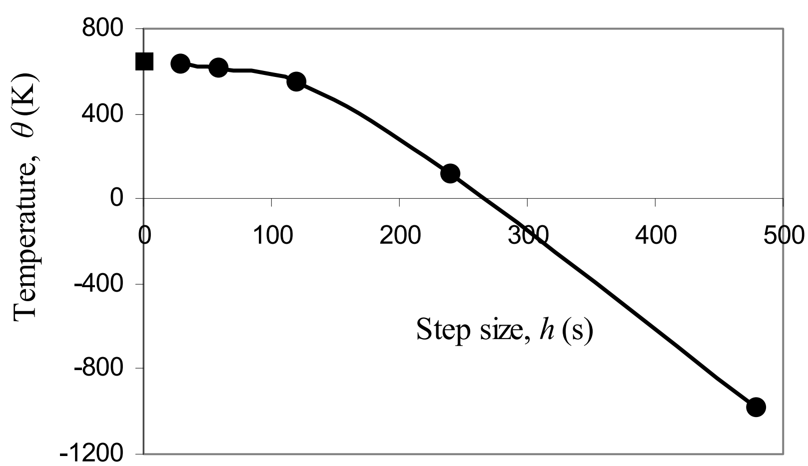


Figure 5 Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.333\times10^{-2}\theta\right)=-0.22067\times10^{-3}t-2.9282 \qquad (4)$$

The solution to this nonlinear equation is

$$\theta = 647.57\,\text{K}$$

It can be seen that Euler's method has large errors. This can be illustrated using the Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}\left(x_{i+1}-x_i\right)+\frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}\left(x_{i+1}-x_i\right)^2+\frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}\left(x_{i+1}-x_i\right)^3+\ldots \qquad (5)$$

$$= y_i + f(x_i,y_i)(x_{i+1}-x_i)+\frac{1}{2!}f'(x_i,y_i)\left(x_{i+1}-x_i\right)^2+\frac{1}{3!}f''(x_i,y_i)\left(x_{i+1}-x_i\right)^3+\ldots \qquad (6)$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

are Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'\left(x_i,y_i\right)}{2!}h^2+\frac{f''\left(x_i,y_i\right)}{3!}h^3+\ldots \qquad (7)$$

The true error hence is approximately proportional to the square of the step size, that is, as the step size is halved, the true error gets approximately quartered. However from Table 1, we see that as the step size gets halved, the true error only gets approximately halved. This is because the true error, being proportioned to the square of the step size, is the local truncation error, that is, error from one point to the next. The global truncation error is however proportional only to the step size as the error keeps propagating from one point to another.

Can one solve a definite integral using numerical methods such as Euler's method of solving ordinary differential equations?

Let us suppose you want to find the integral of a function $f(x)$

$$I=\int_a^b f\left(x\right)dx\,.$$

Both fundamental theorems of calculus would be used to set up the problem so as to solve it as an ordinary differential equation.

The first fundamental theorem of calculus states that if f is a continuous function in the interval [a,b], and F is the antiderivative of f, then

$$\int_a^b f\left(x\right)dx = F\left(b\right)-F\left(a\right)$$

The second fundamental theorem of calculus states that if f is a continuous function in the open interval D, and a is a point in the interval D, and if

$$F\left(x\right)=\int_a^x f\left(t\right)dt$$

then

$$F'(x) = f(x)$$

at each point in D.

Asked to find $\int_a^b f(x)dx$, we can rewrite the integral as the solution of an ordinary differential equation (here is where we are using the second fundamental theorem of calculus)

$$\frac{dy}{dx} = f(x), \; y(a) = 0,$$

where then $y(b)$ (here is where we are using the first fundamental theorem of calculus) will give the value of the integral $\int_a^b f(x)dx$.

Example 4

Find an approximate value of

$$\int_5^8 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

Solution

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3, \; y(5) = 0$$

where $y(8)$ will give the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x, y), \; y(5) = 0$$

The Euler's method equation is

$$y_{i+1} = y_i + f(x_i, y_i)h$$

Step 1

$$i = 0, \; x_0 = 5, \; y_0 = 0$$

$$h = 1.5$$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)h \\ &= 0 + f(5,0) \times 1.5 \\ &= 0 + (6 \times 5^3) \times 1.5 \\ &= 1125 \\ &\approx y(6.5) \end{aligned}$$

Step 2

$$i = 1,\ x_1 = 6.5,\ y_1 = 1125$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 6.5 + 1.5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)h \\ &= 1125 + f(6.5, 1125) \times 1.5 \\ &= 1125 + \left(6 \times 6.5^3\right) \times 1.5 \\ &= 3596.625 \\ &\approx y(8) \end{aligned}$$

Hence

$$\begin{aligned} \int_5^8 6x^3\,dx &= y(8) - y(5) \\ &\approx 3596.625 - 0 \\ &= 3596.625 \end{aligned}$$

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Euler's Method for ordinary differential equations
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Web Site	http://numericalmethods.eng.usf.edu